Poisson-Dirichlet Statistics for the extremes of the two-dimensional discrete Gaussian Free Field

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Abstract. In a previous paper, the authors introduced an approach to prove that the statistics of the extremes of a log-correlated Gaussian field converge to a Poisson-Dirichlet variable at the level of the Gibbs measure at low temperature and under suitable test functions. The method is based on showing that the model admits a one-step replica symmetry breaking in spin glass terminology. This implies Poisson-Dirichlet statistics by general spin glass arguments. In this note, this approach is used to prove Poisson-Dirichlet statistics for the two-dimensional discrete Gaussian free field, where boundary effects demand a more delicate analysis.

1. Introduction

1.1. **The model.** Consider a finite box A of $\mathbb{Z}^2$. The Gaussian free field (GFF) on A with Dirichlet boundary condition is the centered Gaussian field $(\phi_v, v \in A)$ with the covariance matrix

$$G_A(v,v') := E_v\left[\sum_{k=0}^{\tau_A} 1_{v'}(S_k)\right] , \tag{1.1}$$

where $(S_k, k \geq 0)$ is a simple random walk with $S_0 = v$ of law P_v killed at the first exit time of A, τ_A, i.e. the first time where the walk reaches the boundary ∂A. Throughout the paper, for any $A \subset \mathbb{Z}^2$, ∂A will denote the set of vertices in A^c that share an edge with a vertex of A. We will write $\mathbb{P}$ for the law of the Gaussian field and $\mathbb{E}$ for the expectation. For $B \subset A$, we denote the σ-algebra generated by $\{\phi_v, v \in B\}$ by $\mathcal{F}_B$.

We are interested in the case where $A = V_N := \{1, \dots, N\}^2$ in the limit $N \to \infty$. For $0 \leq \delta < 1/2$, we denote by V_N^δ the set of the points of V_N whose distance to the boundary ∂V_N is greater than δN. In this set, the variance of the field diverges logarithmically with N, cf. Lemma 5.2 in the appendix,

$$\mathbb{E}[\phi_v^2] = G_{V_N}(v,v) = \frac{1}{\pi}\log N^2 + O_N(1), \qquad \forall v \in V_N^\delta, \tag{1.2}$$

where $O_N(1)$ will always be a term which is uniformly bounded in N and in $v \in V_N$. (The term $o_N(1)$ will denote throughout a term which goes to 0 as $N \to \infty$ uniformly in all other parameters.) Equation (1.2) follows from the fact that for $v \in V_N^\delta$ and

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$u \in \partial V_N$, $\delta N \leq \|v-u\| \leq \sqrt{2}(1-\delta)N$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{Z}^2$. A similar estimate yields an estimate on the covariance

$$\mathbb{E}[\phi_v \phi_{v'}] = G_{V_N}(v,v') = \frac{1}{\pi}\log\frac{N^2}{\|v-v'\|^2} + O_N(1), \qquad \forall v,v' \in V_N^\delta. \tag{1.3}$$

In view of (1.2) and (1.3), the Gaussian field $(\phi_v, v \in V_N)$ is said to be *log-correlated.* On the other hand, there are many points that are outside V_N^δ (of the order of N^2 points) for which the estimates (1.2) and (1.3) are not correct. Essentially, the closer the points are to the boundary the lesser are the variance and covariance as the simple random walk in (1.1) has a higher probability of exiting V_N early. This decoupling effect close to the boundary complicates the analysis of the extrema of the GFF by comparison with log-correlated Gaussian fields with stationary distribution.

1.2. **Main results.** It was shown by Bolthausen, Deuschel, and Giacomin [7] that the maximum of the GFF in V_N^δ satisfies

$$\lim_{N\to\infty} \frac{\max_{v\in V_N^\delta} \phi_v}{\log N^2} = \sqrt{\frac{2}{\pi}}, \qquad \text{in probability.} \tag{1.4}$$

A comparison argument using Slepian's lemma can be used to extend the result to the whole box V_N. Their technique was later refined by Daviaud [15] who computed the *log-number of high points* in V_N^δ: for $0<\lambda<1$,

$$\lim_{N\to\infty} \frac{1}{\log N^2} \log \#\{v \in V_N^\delta : \phi_v \geq \lambda\sqrt{\frac{2}{\pi}}\log N^2\} = 1-\lambda^2, \qquad \text{in probability.} \tag{1.5}$$

It is a simple exercise to show using the above results that the *free energy* in V_N of the model is given by

$$f(\beta) := \lim_{N\to\infty}\frac{1}{\log N^2}\log\sum_{v\in V_N} e^{\beta\phi_v} = \begin{cases} 1+\frac{\beta^2}{2\pi}, & \text{if } \beta \leq \sqrt{2\pi}, \\ \sqrt{\frac{2}{\pi}}\beta, & \text{if } \beta \geq \sqrt{2\pi}, \end{cases} \qquad \text{a.s. and in } L^1. \tag{1.6}$$

Indeed, there is the clear lower bound $\log\sum_{v\in V_N} e^{\beta\phi_v} \geq \log\sum_{v\in V_N^\delta} e^{\beta\phi_v}$, which can be evaluated using the log-number of high points (1.5) by Laplace's method. The upper bound is obtained using a comparison argument with i.i.d. centered Gaussians.

A striking fact is that the three above results correspond to the expressions for N^2 independent Gaussian variables of variance $\frac{1}{\pi}\log N^2$. In other words, correlations have no effects on the above observables of the extremes. The purpose of the paper is to extend this correspondence to observables related to the Gibbs measure.

To this aim, consider the *normalized Gibbs weights* or *Gibbs measure*

$$\mathcal{G}_{\beta,N}(\{v\}) := \frac{e^{\beta\phi_v}}{Z_N(\beta)}, \qquad v \in V_N,$$

where $Z_N(\beta) := \sum_{v\in V_N} e^{\beta\phi_v}$. We consider the normalized covariance or *overlap*

$$q(v,v') := \frac{\mathbb{E}[\phi_v\phi_{v'}]}{\frac{1}{\pi}\log N^2}, \qquad \forall v,v' \in V_N. \tag{1.7}$$

This is the covariance divided by the dominant term of the variance in the bulk.

In spin glasses, the relevant object to classify the extreme value statistics of strongly correlated variables is the *two-overlap distribution function*

$$x_{\beta,N}(q) := \mathbb{E}\left[\mathcal{G}_{\beta,N}^{\times 2}\left\{q(v,v') \leq q\right\}\right], \qquad 0 \leq q \leq 1. \tag{1.8}$$

The main result shows that the 2D GFF falls within the class of models that exhibit a *one-step replica symmetry breaking* at low temperature.

Theorem 1.1. *For $\beta > \beta_c = \sqrt{2\pi}$,*

$$\lim_{N\to\infty} x_{\beta,N}(r) := \lim_{N\to\infty} \mathbb{E}\left[\mathcal{G}_{\beta,N}^{\times 2}\left\{q(v,v') \leq q\right\}\right] = \begin{cases} \frac{\beta_c}{\beta} & \text{for } 0 \leq r < 1, \\ 1 & \text{for } r = 1. \end{cases}$$

Note that for $\beta \leq \beta_c$, it follows from (1.6) that the overlap is 0 almost surely. The result is the analogue for the 2D GFF of the results obtained by Derrida & Spohn [17] and Bovier & Kurkova [10, 11] for the branching Brownian motion and for GREM-type models. In [4], such a result was proved for a non-hierarchical log-correlated Gaussian field constructed from the multifractal random measure of Bacry & Muzy [5], see also [22] for a closely related model. This type of result was conjectured by Carpentier & Ledoussal [14]. We also remark that Theorem 1.1 shows that at low temperature two points sampled with the Gibbs measure have overlaps 0 or 1. This is consistent with the result of Ding & Zeitouni [19] who showed that the extremal values of GFF are at distance from each other of order one or of order N.

A general method to prove Poisson-Dirichlet statistics for the distribution of the overlaps from the one-step replica symmetry breaking was laid down in [4]. This connection is done via the (now fundamental) Ghirlanda-Guerra identities. Another equivalent approach would be using *stochastic stability* as developed in [1, 2, 3]. The reader is referred to Section 2.3 of [4] where the connection is explained in details for general Gaussian fields. For the sake of conciseness, we simply state the consequence for the 2D GFF.

Consider the product measure $\mathcal{G}_{\beta,N}^{\times s}$ on s *replicas* $(v_1,\dots,v_s) \in V_N^{\times s}$. Let $F : [0,1]^{\frac{s(s-1)}{2}} \to \mathbb{R}$ be a continuous function. Write $F(q_{ll'})$ for the function evaluated at $q_{ll'} := q(v_l, v_{l'})$, $l \neq l'$, for $(v_1,\dots,v_s) \in V_N^{\times s}$. We write $\mathbb{E}\mathcal{G}_{\beta,N}^{\times s}\big(F(q_{ll'})\big)$ for the averaged expectation. Recall that a *Poisson-Dirichlet variable* ξ of parameter α is a random variable on the space of decreasing weights $\boldsymbol{s} = (s_1, s_2, \dots)$ with $1 \geq s_1 \geq s_2 \geq \dots \geq 0$ and $\sum_i s_i \leq 1$ which has the same law as $\Big(\eta_i / \sum_j \eta_j, i \in \mathbb{N}\Big)_\downarrow$ where $\downarrow$ stands for the decreasing rearrangement and $\eta = (\eta_i, i \in \mathbb{N})$ are the atoms of a Poisson random measure on $(0,\infty)$ of intensity measure $s^{-\alpha-1}\, ds$.

The theorem below is a direct consequence of the Theorem 1.1, the differentiability of the free energy (1.6) as well as Corollary 2.5 and Theorem 2.6 of [4].

Theorem 1.2. *Let $\beta > \beta_c$ and $\xi = (\xi_k, k \in \mathbb{N})$ be a Poisson-Dirichlet variable of parameter β_c/β. Denote by E the expectation with respect to ξ. For any continuous function $F : [0,1]^{\frac{s(s-1)}{2}} \to \mathbb{R}$ of the overlaps of s replicas:*

$$\lim_{N\to\infty} \mathbb{E}\left[\mathcal{G}_{\beta,N}^{\times s}\left(F(q_{ll'})\right)\right] = E\left[\sum_{k_1\in\mathbb{N},\dots,k_s\in\mathbb{N}} \xi_{k_1}\dots\xi_{k_s}\ F(\delta_{k_l k_{l'}})\right].$$

The above is one of the few rigorous results known on the Gibbs measure of log-correlated fields at low temperature. Theorem 1.2 is a step closer to the conjecture of Duplantier, Rhodes, Sheffield & Vargas (see Conjecture 11 in [20] and Conjecture 6.3 in [30]) that the Gibbs measure, as a random probability measure on V_N, should be atomic in the limit with the size of the atoms being Poisson-Dirichlet. Theorem 1.2 falls short of the full conjecture because only test-functions of the overlaps are considered. Finally, it is expected that the Poisson-Dirichlet statistics emerging here is related to the Poissonian statistics of the thinned extrema of the 2D GFF proved by Biskup & Louidor in [6] based on the convergence of the maximum established by Bramson, Ding & Zeitouni [12]. To recover the Gibbs measure from the extremal process, some properties of the cluster of points near the maxima must be known.

The rest of this paper is dedicated to the proof of Theorem 1.1. In Section 2, a generalized version of the GFF (whose variance is scale-dependent) is introduced. It is a kind of non-hierarchical GREM and is related to a model studied by Fyodorov & Bouchaud in [23]. The proof of Theorem 1.1 is given in Section 3. It relates the overlap distribution of the 2D GFF to the free energy of the generalized GFF. The free energy of the generalized GFF needed in the proof is computed in Section 4.

2. The multiscale decomposition and a generalized GFF

In this section, we construct a Gaussian field from the GFF whose variance is scale-dependent. The construction uses a multiscale decomposition along each vertex. The construction is analogous to a *Generalized Random Energy Model* of Derrida [16], but where correlations are non-hierarchical. Here, only two different values of the variance will be needed though the construction can be directly generalized to any finite number of values.

Consider $0 < t < 1$. We assume to simplify the notation that N^{1-t} is an even integer and that N^t divides N. The case of general t's can also be done by making trivial corrections along the construction.

For $v \in V_N$, we write $[v]_t$ for the unique box with N^{1-t} points on each side and centered at v. If $[v]_t$ is not entirely contained in V_N, we take the convention that $[v]_t$ is the intersection of the square box with V_N. For $t = 1$, take $[v]_1 = v$. The σ-algebra $\mathcal{F}_{[v]_t^c}$ is the σ-algebra generated by the field outside $[v]_t$. We define

$$\phi_{[v]_t} := \mathbb{E}\left[\phi_v \;\middle|\; \mathcal{F}_{[v]_t^c}\right] = \mathbb{E}\left[\phi_v \;\middle|\; \mathcal{F}_{\partial[v]_t}\right] \ ,$$

where the second equality holds by the Markov property of the Gaussian free field, see Lemma 5.1. Clearly, for any $v \in V_N$, the random variable $\phi_{[v]_t}$ is Gaussian. Moreover, by Lemma 5.1,

$$\phi_{[v]_t} = \sum_{u \in \partial[v]_t} p_{t,v}(u)\phi_u \ , \tag{2.1}$$

where $p_{t,v}(u) = P_v(S_{\tau_{[v]_t}} = u)$ is the probability that a simple random walk starting at v hits u at the first exit time of $[v]_t$.

The following *multiscale decomposition* holds trivially

$$\phi_v = \phi_{[v]_t} + \left(\phi_v - \phi_{[v]_t}\right) \ . \tag{2.2}$$

The decomposition suggests the following scale-dependent perturbation of the field. For $0 < \alpha < 1$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2) \in \mathbb{R}_+^2$, consider for $v \in V_N$,

$$\psi_v := \sigma_1 \phi_{[v]_\alpha} + \sigma_2 \left(\phi_v - \phi_{[v]_\alpha}\right) \ . \tag{2.3}$$

The Gaussian field $(\psi_v, v \in V_N)$ will be called the $(\alpha, \boldsymbol{\sigma})$-GFF on V_N.

To control the boundary effects, it is necessary to consider the field in a box slightly smaller than V_N. For $\rho \in (0,1)$, let

$$A_{N,\rho} := \{v \in V_N : d_1(v, \partial V_N) \geq N^{1-\rho}\} \ , \tag{2.4}$$

where $d_1(v,B) := \inf\{\|v-u\| \, ; \, u \in B\}$ for any set $B \subset \mathbb{Z}^2$. We always take $\rho < \alpha$ so that $[v]_\alpha$ is contained in V_N for any $v \in A_{N,\rho}$. We write $\mathcal{G}^{(\alpha,\boldsymbol{\sigma})}_{\beta,N,\rho}(\cdot)$ for the Gibbs measure of $(\alpha, \boldsymbol{\sigma})$-GFF restricted to $A_{N,\rho}$

$$\mathcal{G}^{(\alpha,\boldsymbol{\sigma})}_{\beta,N,\rho}(\{v\}) := \frac{e^{\beta\psi_v}}{Z^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta)}, \qquad v \in A_{N,\rho},$$

where $Z^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta) := \sum_{v \in A_{N,\rho}} e^{\beta\psi_v}$.

The associated free energy is given by

$$f^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta) := \frac{1}{\log N^2} \log Z^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta), \qquad \forall \beta > 0.$$

(Note that $\log \# A_{N,\rho} = (1 + o_N(1)) \log N^2$.) Its L_1-limit is a central quantity needed to apply Bovier-Kurkova technique. This limit is better expressed in terms of the free energy of the REM model consisting of N^2 i.i.d. Gaussian variables of variance $\frac{\sigma^2}{\pi} \log N^2$:

$$f(\beta; \sigma^2) := \begin{cases} 1 + \frac{\beta^2\sigma^2}{2\pi}, & \text{if } \beta \leq \beta_c(\sigma^2) := \frac{\sqrt{2\pi}}{\sigma}, \\ \sqrt{\frac{2}{\pi}}\sigma\beta, & \text{if } \beta \geq \beta_c(\sigma^2). \end{cases} \tag{2.5}$$

Theorem 2.1. *Fix $\alpha \in (0,1)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2) \in \mathbb{R}^2_+$ and let $V_{12} := \sigma_1^2\alpha + \sigma_2^2(1-\alpha)$. Then, for any $\rho < \alpha$, and for all $\beta > 0$*

$$\lim_{N\to\infty} f^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta) = f^{(\alpha,\boldsymbol{\sigma})}(\beta) := \begin{cases} f(\beta; V_{12}), & \text{if } \sigma_1 \leq \sigma_2, \\ \alpha f(\beta;\sigma_1^2) + (1-\alpha) f(\beta; \sigma_2^2), & \text{if } \sigma_1 \geq \sigma_2, \end{cases} \tag{2.6}$$

where the convergence holds almost surely and in L^1.

Note that the limit does not depend on ρ.

3. Proof of Theorem 1.1

3.1. **The Gibbs measure close to the boundary.** The first step in the proof of Theorem 1.1 is to show that points close to the boundary do not carry any weight in the Gibbs measure of the GFF in V_N. The result would not necessarily hold if we considered instead the outside of V_N^δ which is much larger than the outside of $A_{N,\rho}$.

Lemma 3.1. *For any $\rho > 0$,*

$$\lim_{N\to\infty} \mathcal{G}_{\beta,N}(A^c_{N,\rho}) = 0, \qquad \text{in } \mathbb{P}\text{-probability.} \tag{3.1}$$

Before turning to the proof, we claim that the lemma implies that, for any $r \in [0,1]$ and $\rho \in (0,1)$,

$$\lim_{N\to\infty} \left| x_{\beta,N}(r) - x_{\beta,N,\rho}(r) \right| = 0 \ , \tag{3.2}$$

where

$$x_{\beta,N,\rho}(r) := \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\{q(v,v') \le r\}, \qquad r \in [0,1] \ . \tag{3.3}$$

is the two-overlap distribution of the Gibbs measure of the GFF $(\phi_v, v \in V_N)$ restricted to $A_{N,\rho}$

$$\mathcal{G}_{\beta,N,\rho}(\{v\}) := \frac{\mathrm{e}^{\beta\phi_v}}{Z_{N,\rho}(\beta)}, \qquad v \in A_{N,\rho},$$

for $Z_{N,\rho}(\beta) := \sum_{v\in A_{N,\rho}} \mathrm{e}^{\beta\phi_v}$. Indeed, introducing an auxiliary term

$$\begin{aligned}\left|x_{\beta,N}(r) - x_{\beta,N,\rho}(r)\right| &\le \left|\mathbb{E}\mathcal{G}^{\times 2}_{\beta,N}\{q(v,v') \le r\} - \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N}\{q(v,v') \le r; v,v' \in A_{N,\rho}\}\right| \\ &\quad + \left|\mathbb{E}\mathcal{G}^{\times 2}_{\beta,N}\{q(v,v') \le r; v,v' \in A_{N,\rho}\} - \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\{q(v,v') \le r\}\right| \ .\end{aligned}$$

The first term is smaller than $2\ \mathbb{E}\mathcal{G}_{\beta,N}(A^c_{N,\rho})$. The second term equals

$$\begin{aligned}&\mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\{q(v,v') \le r\} - \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N}\{q(v,v') \le r; v,v' \in A_{N,\rho}\} \\ &\qquad = \mathbb{E}\left[\frac{\mathcal{G}^{\times 2}_{\beta,N}\{q(v,v') \le r; v,v' \in A_{N,\rho}\}}{\mathcal{G}^{\times 2}_{\beta,N}\{v,v' \in A_{N,\rho}\}}\left(1 - \mathcal{G}^{\times 2}_{\beta,N}\{v,v' \in A_{N,\rho}\}\right)\right],\end{aligned}$$

which is also smaller than $2\ \mathbb{E}\mathcal{G}_{\beta,N}(A^c_{N,\rho})$. Lemma 3.1 then implies (3.2) as claimed.

Proof of Lemma 3.1. Let $\epsilon > 0$ and $\lambda > 0$. The probability can be split as follows

$$\begin{aligned}\mathbb{P}\left(\mathcal{G}_{\beta,N}(A^c_{N,\rho}) > \epsilon\right) &\le \mathbb{P}\left(\mathcal{G}_{\beta,N}(A^c_{N,\rho}) > \epsilon, \left|\frac{1}{\log N^2}\log Z_N(\beta) - f(\beta)\right| \le \lambda\right) \\ &\quad + \mathbb{P}\left(\left|\frac{1}{\log N^2}\log Z_N(\beta) - f(\beta)\right| > \lambda\right) \ ,\end{aligned}$$

where $f(\beta)$ is defined in (1.6). The second term converges to zero by (1.6). The first term is smaller than

$$\mathbb{P}\left(\frac{1}{\log N^2}\log\sum_{v\in A^c_{N,\rho}}\exp\beta\phi_v > f(\beta) - \lambda + \frac{\log\epsilon}{\log N^2}\right) . \tag{3.4}$$

Since the free energy is a Lipschitz function of the variables ϕ_v, see e.g. Theorem 2.2.4 in [31], the free energy self-averages, that is for any $t > 0$

$$\lim_{N\to\infty}\mathbb{P}\left(\left|\frac{1}{\log N^2}\log\sum_{v\in A^c_{N,\rho}}\exp\beta\phi_v - \frac{1}{\log N^2}\mathbb{E}\left[\log\sum_{v\in A^c_{N,\rho}}\exp\beta\phi_v\right]\right| \ge t\right) = 0 \ .$$

To conclude the proof, it remains to show that for some $C < 1$ (independent of N but dependent on ρ)

$$\limsup_{N\to\infty}\frac{1}{\log N^2}\mathbb{E}\left[\log\sum_{v\in A^c_{N,\rho}}\exp\beta\phi_v\right] < Cf(\beta). \tag{3.5}$$

Note that by Lemma 5.1, the maximal variance of ϕ_v in V_N is $\frac{1}{\pi}\log N^2 + O_N(1)$. Pick $(g_v, v \in A^c_{N,\rho})$ independent centered Gaussians (and independent of $(\phi_v)_{v\in A^c_{N,\rho}}$) with variance given by $\mathbb{E}[g_v^2] = \frac{1}{\pi}\log N^2 + O_N(1) - \mathbb{E}[\phi_v^2]$. Jensen's inequality applied to the Gibbs measure implies that $\mathbb{E}[\log\sum_{v\in A^c_{N,\rho}}\exp\beta(\phi_v + g_v)] \ge \mathbb{E}[\log\sum_{v\in A^c_{N,\rho}}\exp\beta\phi_v]$. Moreover, by a standard comparison argument (see Lemma 5.3 in the Appendix),

$\mathbb{E}[\log \sum_{v\in A^c_{N,\rho}} \exp \beta(\phi_v + g_v)]$ is smaller than the expectation for i.i.d. variables with identical variances. The two last observations imply that

$$\frac{1}{\log N^2}\mathbb{E}\left[\log \sum_{v\in A^c_{N,\rho}} \exp \beta\phi_v\right] \leq \frac{1}{\log N^2}\mathbb{E}\left[\log \sum_{v\in A^c_{N,\rho}} \exp \beta\widetilde{\phi}_v\right],$$

where $(\widetilde{\phi}_v, v\in A^c_{N,\rho})$ are i.i.d. centered Gaussians of variance $\frac{1}{\pi}\log N^2 + O_N(1)$. Since $\#A^c_{N,\rho} = N^2 - |A_{N,\rho}| = 4N^{2-\rho}(1+o_N(1))$, the free energy of these i.i.d. Gaussians in the limit $N\to\infty$ is given by (2.5)

$$\lim_{N\to\infty}\frac{1}{\log 4N^{2-\rho}}\mathbb{E}\left[\log \sum_{v\in A^c_{N,\rho}} \exp \beta\widetilde{\phi}_v\right] = \begin{cases} 1+\frac{\beta^2}{2\pi}\left(1-\frac{\rho}{2}\right)^{-1}, & \beta < \sqrt{2\pi}\left(1-\frac{\rho}{2}\right)^{1/2}, \\ \sqrt{\frac{2}{\pi}}\left(1-\frac{\rho}{2}\right)^{-1/2}\beta, & \beta \geq \sqrt{2\pi}\left(1-\frac{\rho}{2}\right)^{1/2}. \end{cases}$$

The last two equations then imply

$$\limsup_{N\to\infty}\frac{1}{\log N^2}\mathbb{E}\left[\log \sum_{v\in A^c_{N,\rho}} \exp \beta\phi_v\right] \leq \begin{cases} \left(1-\frac{\rho}{2}\right)+\frac{\beta^2}{2\pi}, & \beta < \sqrt{2\pi}\left(1-\frac{\rho}{2}\right)^{1/2}, \\ \sqrt{\frac{2}{\pi}}\left(1-\frac{\rho}{2}\right)^{1/2}\beta, & \beta \geq \sqrt{2\pi}\left(1-\frac{\rho}{2}\right)^{1/2}. \end{cases}$$

It is then straightforward to check that, for every β, the right side is strictly smaller than $f(\beta)$ as claimed. □

3.2. **An adaptation of the Bovier-Kurkova technique.** Theorem 1.1 follows from Equation (3.2) and

Proposition 3.2. *For $\beta > \beta_c = \sqrt{2\pi}$,*

$$\lim_{\rho\to 0}\lim_{N\to\infty} x_{\beta,N,\rho}(r) = \begin{cases} \frac{\beta_c}{\beta}, & \text{for } 0\leq r<1, \\ 1, & \text{for } r=1. \end{cases}$$

Proof. Without loss of generality, we suppose that $\lim_{\rho\to 0}\lim_{N\to\infty} x_{\beta,N,\rho} = x_\beta$ in the sense of weak convergence. Uniqueness of the limit x_β will then ensure the convergence for the whole sequence by compactness. Note also that by right-continuity and monotonicity of x_β, it suffices to show

$$\int_\alpha^1 x_\beta(r)dr = \frac{\beta_c}{\beta}(1-\alpha), \qquad \text{for a dense set of } \alpha\text{'s in } [0,1]. \tag{3.6}$$

We can choose a dense set of α such that none of them are atoms of x_β, that is $x_\beta(\alpha) - x_\beta(\alpha^-) = 0$.

Now recall Theorem 2.1. Pick $\boldsymbol{\sigma} = (1, 1+u)$ for some parameter $|u|\leq 1$. Since $\beta > \sqrt{2\pi}$, u can be taken small enough so that β is larger than the critical β's of the limit. The goal is to establish the following equality:

$$\int_\alpha^1 x_\beta(r)dr = \lim_{\rho\to 0}\lim_{N\to\infty}\frac{\pi}{\beta^2}\frac{\partial}{\partial u} f^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta)\Big|_{u=0}. \tag{3.7}$$

The conclusion follows from this equality. Indeed, by construction, the function $u\mapsto f^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta)$ is convex. In particular, the limit of the derivatives is the derivative of the limit at any point of differentiability. Therefore, a straightforward calculation from (2.6) with $\sigma_1 = 1$ and $\sigma_2 = 1 + u$ gives:

$$\lim_{N\to\infty} \frac{\pi}{\beta^2}\frac{\partial}{\partial u} f_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta) = \begin{cases} \frac{\sqrt{2\pi}}{\beta}\frac{(1-\alpha)(1+u)}{\sqrt{\alpha+(1-\alpha)(1+u)^2}}, & \text{if } u>0,\\ \frac{\sqrt{2\pi}}{\beta}(1-\alpha), & \text{if } u<0. \end{cases} \tag{3.8}$$

This gives (3.6) at $u = 0$.

We introduce the notation for the *overlap at scale* α:

$$q_\alpha(v,v') := \frac{1}{\frac{1}{\pi}\log N^2}\mathbb{E}\left[\left(\phi_v - \phi_{[v]_\alpha}\right)\left(\phi_{v'} - \phi_{[v']_\alpha}\right)\right], \tag{3.9}$$

Equality (3.7) is proved via two identities:

$$\int_\alpha^1 x_{\beta,N,\rho}(r)dr = (1-\alpha) - \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\big[q(v,v')-\alpha; q(v,v')\geq \alpha\big], \tag{3.10}$$

$$\frac{\pi}{\beta^2}\frac{\partial}{\partial u} f_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta)\Big|_{u=0} = \mathbb{E}\mathcal{G}_{\beta,N,\rho}\big[q_\alpha(v,v)\big] - \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\big[q_\alpha(v,v'); v'\in[v]_\alpha\big]\ . \tag{3.11}$$

The first identity holds since by Fubini's theorem

$$\begin{aligned}\int_\alpha^1 x_{\beta,N,\rho}(r)dr &= \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\left[\int_\alpha^1 1_{\{r\geq q(v,v')\}}dr\right]\\ &= \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\big[1-\alpha; q(v,v')<\alpha\big] + \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\big[1-q(v,v'); q(v,v')\geq\alpha\big]\ .\end{aligned}$$

For the second identity, direct differentiation gives

$$\frac{\pi}{\beta^2}\frac{\partial}{\partial u} f_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta)\Big|_{u=0} = \frac{1}{\frac{1}{\pi}\log N^2}\mathbb{E}\mathcal{G}_{\beta,N,\rho}\big[\phi_v - \phi_{[v]_\alpha}\big]\ .$$

The identity is then obtained by Gaussian integration by parts.

To prove (3.7), we need to relate the overlap at scale α with the overlap as well as the event $\{q(v,v')\geq\alpha\}$ with the event $\{v'\in[v]_\alpha\}$. This is slightly complicated by the boundary effect present in GFF. The equality in the limit $N\to\infty$ between the first terms of (3.10) and (3.11) is easy. Because $(\phi_u - \mathbb{E}[\phi_u|\mathcal{F}_{[v]^c_\alpha}], u\in[v]_\alpha)$ has the law of a GFF in $[v]_\alpha$, it follows from Lemma 5.2 that

$$\mathbb{E}\big[(\phi_v - \phi_{[v]_\alpha})^2\big] = \frac{(1-\alpha)}{\pi}\log N^2 + O_N(1)\ .$$

Therefore, we have for $v\in A_{N,\rho}$

$$\lim_{N\to\infty}\mathbb{E}\mathcal{G}_{\beta,N,\rho}\left[q_\alpha(v,v)\right] = 1-\alpha\ .$$

It remains to establish the equality between the second terms of (3.10) and (3.11). Here, a control of the boundary effect is necessary. The following observation is useful to relate the overlaps and the distances: if $v,v'\in A_{N,\rho}$, Lemma 5.2 gives

$$1-\rho-\frac{\log\|v-v'\|^2}{\log N^2} + o_N(1) \leq q(v,v') \leq 1 - \frac{\log\|v-v'\|^2}{\log N^2} + o_N(1)\ . \tag{3.12}$$

On one hand, the right inequality proves the following implication

$$q(v,v')\geq\alpha+\varepsilon \text{ for some } \varepsilon>0 \Longrightarrow \|v-v'\|^2\leq cN^{2(1-\alpha-\varepsilon)}\ , \tag{3.13}$$

for some constant c independent of N and ρ. On the other hand, the left inequality gives:

$$v'\in[v]_\alpha \Longrightarrow q(v,v')\geq\alpha-2\rho. \tag{3.14}$$

Using this, we show

$$\Delta_1(N,\rho) := \left|\mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\left[q(v,v')-\alpha; q(v,v')\geq \alpha\right] - \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\left[q_\alpha(v,v'); q(v,v')\geq \alpha\right]\right| \to 0\ , \tag{3.15}$$
$$\Delta_2(N,\rho) := \left|\mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\left[q_\alpha(v,v'); q(v,v')\geq \alpha\right] - \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\left[q_\alpha(v,v'); v'\in [v]_\alpha\right]\right| \to 0\ ,$$

in the limit $N\to\infty$ and $\rho\to 0$. Let $\varepsilon>0$. Remark that

$$0\leq \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\left[q(v,v')-\alpha; q(v,v')\geq \alpha\right] - \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\left[q(v,v')-\alpha; q(v,v')\geq \alpha+\varepsilon\right]\leq \varepsilon\ . \tag{3.16}$$

To establish the equality of the overlaps on the event $\{q(v,v')\geq\alpha+\varepsilon\}$, consider the decomposition,

$$\begin{aligned}&\mathbb{E}\left[\left(\phi_v-\phi_{[v]_\alpha}\right)\left(\phi_{v'}-\phi_{[v']_\alpha}\right)\right] = \\ &\mathbb{E}\left[\left(\phi_v-\mathbb{E}[\phi_v|\mathcal{F}_{[v']^c_\alpha}]\right)\left(\phi_{v'}-\phi_{[v']_\alpha}\right)\right] + \mathbb{E}\left[\left(\mathbb{E}[\phi_v|\mathcal{F}_{[v']^c_\alpha}]-\phi_{[v]_\alpha}\right)\left(\phi_{v'}-\phi_{[v']_\alpha}\right)\right]\ .\end{aligned} \tag{3.17}$$

On the event $\{q(v,v')\geq\alpha+\varepsilon\}$, (3.13) implies $\|v-v'\|^2\leq cN^{2(1-\alpha-\varepsilon)}$. Therefore, the first term of the right side of (3.17) is by Lemma 5.2

$$\mathbb{E}\left[\left(\phi_v-\mathbb{E}[\phi_v|\mathcal{F}_{[v']^c_\alpha}]\right)\left(\phi_{v'}-\phi_{[v']_\alpha}\right)\right] = \frac{2}{\pi}\log\frac{N^{(1-\alpha)}}{\|v-v'\|}+O_N(1)\ . \tag{3.18}$$

The second term is negligible. Indeed, by Cauchy-Schwarz inequality, it suffices to prove that

$$\mathbb{E}\left[\left(\mathbb{E}[\phi_v|\mathcal{F}_{[v']^c_\alpha}]-\phi_{[v]_\alpha}\right)^2\right] = O_N(1)\ . \tag{3.19}$$

For this, write $\widetilde{B}$ for the box $[v]_\alpha\cap[v']_\alpha$. We have

$$\phi_v-\phi_{[v]_\alpha} = (\phi_v-\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}]) + (\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}]-\phi_{[v]_\alpha})\ .$$

Since $\phi_v-\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}]$ is independent of $\mathcal{F}_{\widetilde{B}^c}$ and $\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}]-\phi_{[v]_\alpha}$ is $\mathcal{F}_{\widetilde{B}^c}$-measurable (observe that $\mathcal{F}_{\widetilde{B}^c}\supset\mathcal{F}_{[v]^c_\alpha}$), we get

$$\mathbb{E}[(\phi_v-\phi_{[v]_\alpha})^2] = \mathbb{E}[(\phi_v-\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}])^2] + \mathbb{E}[(\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}]-\phi_{[v]_\alpha})^2]\ .$$

Moreover, $\mathbb{E}[(\phi_v-\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}])^2]$ and $\mathbb{E}[(\phi_v-\phi_{[v]_\alpha})^2]$ are both equal to $\frac{1-\alpha}{\pi}\log N^2+O_N(1)$ by Lemma 5.2 and the fact that distances of v to vertices in $\partial\widetilde{B}$ and $\partial[v]_\alpha$ are both proportional to $N^{1-\alpha}$. Therefore $\mathbb{E}[(\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}]-\phi_{[v]_\alpha})^2]=O_N(1)$. The same argument with $\phi_{[v]_\alpha}$ replaced by $\mathbb{E}[\phi_v|\mathcal{F}_{[v']^c_\alpha}]$ shows that $\mathbb{E}[(\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}]-\mathbb{E}[\phi_v|\mathcal{F}_{[v']^c_\alpha}])^2]=O_N(1)$. The two equalities imply (3.19). Equations (3.18) and (3.19) give

$$q_\alpha(v,v') = 1-\alpha-\frac{\log\|v-v'\|^2}{\log N^2}+o_N(1),\qquad \text{on } \{q(v,v')\geq\alpha+\varepsilon\}. \tag{3.20}$$

Equations (3.12), (3.16) and (3.20) yield $\Delta_1(N,\rho)\to 0$ in the limit $N\to\infty$, $\rho\to 0$ and $\varepsilon\to 0$.

For $\Delta_2(N,\rho)$, let $\varepsilon'>2\rho$. For $v'\in[v]_\alpha$, (3.14) implies $q(v,v')\geq\alpha-2\rho$. On the other hand, by (3.13), $q(v,v')\geq\alpha+\varepsilon'$ implies $v'\in[v]_\alpha$. These two observations give the estimate

$$\Delta_2(N,\rho)\leq \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\left[q_\alpha(v,v'); q(v,v')\in[\alpha-\varepsilon',\alpha+\varepsilon']\right]\ .$$

The right side is clearly smaller than

$$x_{\beta,N,\rho}(\alpha+\varepsilon')-x_{\beta,N,\rho}(\alpha-\varepsilon')\ .$$

Under the successive limits $N \to \infty$, $\rho \to 0$, then $\varepsilon' \to 0$, the right side becomes $x_\beta(\alpha) - x_\beta(\alpha-)$. This is zero since α was chosen not to be an atom of x_β. $\square$

4. The free energy of the $(\alpha, \boldsymbol{\sigma})$-GFF: proof of Theorem 2.1

The computation of the free energy of the $(\alpha, \boldsymbol{\sigma})$-GFF is divided in two steps. First, an upper bound is found by comparing the field ψ in $A_{N,\rho}$ with a "non-homogeneous" GREM having the same free energy as a standard 2-level GREM. Second, we get a matching lower bound using the trivial inequality $f_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta) \geq \frac{1}{\log N^2} \log \sum_{v \in V_N^\delta} \mathrm{e}^{\beta \psi_v}$. The limit of the right term is computed following the method of Daviaud [15].

4.1. **Proof of the upper bound.** For conciseness, we only prove the case $\sigma_1 \geq \sigma_2$, by a comparison argument with a 2-level GREM. The case $\sigma_1 \leq \sigma_2$ is done similarly by comparing with a REM. The comparison argument will have to be done in two steps to account for boundary effects.

Divide the set $A_{N,\rho}$ into square boxes of side-length $N^{1-\alpha}/100$. (The factor $1/100$ is a choice. We simply need these boxes to be smaller than the neighborhoods $[v]_\alpha$, yet of the same order of length in N.) Pick the boxes in such a way that each $v \in A_{N,\rho}$ belongs to one and only one of these boxes. The collection of boxes is denoted by $\mathcal{B}_\alpha$ and $\partial\mathcal{B}_\alpha$ denotes $\bigcup_{B \in \mathcal{B}_\alpha} \partial B$. For $v \in A_{N,\rho}$, we write $B(v)$ for the box of $\mathcal{B}_\alpha$ to which v belongs. For $B \in \mathcal{B}_\alpha$, denote by $\widetilde{B} \supset B$ the square box given by the intersections of all $[u]_\alpha$, $u \in B$, see figure 1. Remark that the side-length of $\widetilde{B}$ is $cN^{1-\alpha}$, for some constant c. For short, write $\phi_{\widetilde{B}} := \mathbb{E}[\phi_{v_B} | \mathcal{F}_{\widetilde{B}^c}]$ where v_B is the center of the box B. The idea in constructing the GREM is to associate to each point $v \in B$ the same contribution at scale α, namely $\phi_{\widetilde{B}}$. One problem is that $\phi_{\widetilde{B}}$ will not have the same variance for every B since it depends on the distance to the boundary. This is the reason why the comparison will need to be done in two steps.

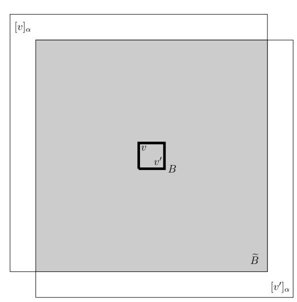

Figure 1. The box $B \in \mathcal{B}_\alpha$ and the corresponding box $\widetilde{B}$ which is the intersection of all the neighborhoods $[v]_\alpha$, $v \in B$.

First, consider the hierarchical Gaussian field $(\widetilde{\psi}_v, v \in A_{N,\rho})$:

$$\widetilde{\psi}_v = g_{B(v)}^{(1)} + g_v^{(2)}, \tag{4.1}$$

where $(g_v^{(2)}, v \in A_{N,\rho})$ are independent centered Gaussians (also independent from $(g_B^{(1)}, B \in \mathcal{B}_\alpha)$) with variance

$$\mathbb{E}[(g_v^{(2)})^2] = \mathbb{E}[\psi_v^2] - \mathbb{E}[(g_{B(v)}^{(1)})^2] \ .$$

This ensures that $\mathbb{E}[\psi_v^2] = \mathbb{E}[\widetilde{\psi}_v^2]$ for all $v \in A_{N,\rho}$. The variables $(g_B^{(1)}, B \in \mathcal{B}_\alpha)$ are also independent centered Gaussians with variance chosen to be $\sigma_1^2 \mathbb{E}[\phi_{\widetilde{B}}^2] + C$ for some constant $C \in \mathbb{R}$ independent of B in $\mathcal{B}_\alpha$ and independent of N. The next lemma ensures that

$$\mathbb{E}[\psi_v \psi_{v'}] \geq \mathbb{E}[\widetilde{\psi}_v \widetilde{\psi}_{v'}] \ . \tag{4.2}$$

Lemma 4.1. *Consider the field $(\psi_v, v \in A_{N,\rho})$ as in (2.3). Then $\mathbb{E}[\psi_v \psi_{v'}] \geq 0$. Moreover, if v and v' both belong to $B \in \mathcal{B}_\alpha$, then*

$$\mathbb{E}[\psi_v \psi_{v'}] \geq \sigma_1^2 \mathbb{E}[\phi_{\widetilde{B}}^2] + C \ ,$$

for some constant $C \in \mathbb{R}$ independent of N.

Proof. For the first assertion, write

$$\psi_v = (\sigma_1 - \sigma_2)\phi_{[v]_\alpha} + \sigma_2 \phi_v \ .$$

The representation $\phi_{[v]_\alpha} = \sum_{u \in \partial [v]_\alpha} p_{\alpha,v}(u)\ \phi_u$ of Lemma 5.1 and the fact that $\sigma_1 > \sigma_2$ imply that $\mathbb{E}[\psi_v \psi_v'] \geq 0$ since the field ϕ is positively correlated by (1.1).

Suppose now that $v, v' \in B$ where $B \in \mathcal{B}_\alpha$. The covariance can be written as

$$\begin{aligned} \mathbb{E}[\psi_v \psi_{v'}] &= \sigma_1^2 \mathbb{E}\left[\phi_{[v]_\alpha}\phi_{[v']_\alpha}\right] + \sigma_2^2 \mathbb{E}\left[(\phi_v - \phi_{[v]_\alpha})(\phi_{v'} - \phi_{[v']_\alpha})\right] \\ &\quad + \sigma_1\sigma_2 \mathbb{E}\left[\phi_{[v]_\alpha}(\phi_{v'} - \phi_{[v']_\alpha})\right] + \sigma_1\sigma_2 \mathbb{E}\left[\phi_{[v']_\alpha}(\phi_v - \phi_{[v]_\alpha})\right] \ . \end{aligned} \tag{4.3}$$

We first prove that the last two terms of (4.3) are positive. By Lemma 5.1, we can write $\phi_{[v]_\alpha} = \sum_{u \in \partial [v]_\alpha} p_{\alpha,v}(u)\ \phi_u$. Note that the vertices u that are in $[v']_\alpha^c$ will not contribute to the covariance $\mathbb{E}\left[\phi_{[v]_\alpha}(\phi_{v'} - \phi_{[v']_\alpha})\right]$ by conditioning. Thus

$$\begin{aligned} \mathbb{E}\left[\phi_{[v]_\alpha}(\phi_{v'} - \phi_{[v']_\alpha})\right] &= \sum_{u \in \partial [v]_\alpha \cap [v']_\alpha} p_{\alpha,v}(u)\ \mathbb{E}\left[\phi_u(\phi_{v'} - \phi_{[v']_\alpha})\right] \\ &= \sum_{u \in \partial [v]_\alpha \cap [v']_\alpha} p_{\alpha,v}(u)\ \mathbb{E}\left[(\phi_u - \mathbb{E}[\phi_u | \mathcal{F}_{[v']_\alpha^c}])(\phi_{v'} - \mathbb{E}[\phi_{v'} | \mathcal{F}_{[v']_\alpha^c}])\right] \ . \end{aligned}$$

Lemma 5.2 ensures that the correlation in the sum are positive.

For the first term of (4.3), the idea is to show that $\phi[v]_\alpha$ and $\phi_{\widetilde{B}}$ are close in the L^2-sense. The same argument used to prove (3.19) shows that

$$\mathbb{E}\left[\left(\phi_{[v]_\alpha} - \mathbb{E}[\phi_v | \mathcal{F}_{\widetilde{B}^c}]\right)^2\right] = O_N(1) \ . \tag{4.4}$$

Moreover, since v and v_B are also at a distance smaller than $N^{1-\alpha}/100$ from each other, Lemma 12 in [7] implies that

$$\mathbb{E}\left[\left(\phi_{\widetilde{B}} - \mathbb{E}[\phi_v | \mathcal{F}_{\widetilde{B}^c}]\right)^2\right] = O_N(1) \ . \tag{4.5}$$

Equations (4.4) and (4.5) give $\mathbb{E}[(\phi_{\widetilde{B}} - \phi_{[v]_\alpha})^2] = O_N(1)$ and similarly for v'. All the above sum up to

$$\sigma_1^2 \mathbb{E}\left[\phi_{[v]_\alpha}\phi_{[v']_\alpha}\right] = \sigma_1^2 \mathbb{E}[\phi_{\widetilde{B}}^2] + O_N(1) \ . \tag{4.6}$$

It remains to show that the second term of (4.3) is greater than $O_N(1)$. Since $\phi_{[v]_\alpha}$ and $\phi_{[v']_\alpha}$ are $\mathcal{F}_{\widetilde{B}^c}$-measurable by definition of the box $\widetilde{B}$, we have the decomposition

$$\begin{aligned} \mathbb{E}\left[(\phi_v - \phi_{[v]_\alpha})(\phi_{v'} - \phi_{[v']_\alpha})\right] &= \mathbb{E}[(\phi_v - \mathbb{E}[\phi_v | \mathcal{F}_{\widetilde{B}^c}])(\phi_{v'} - \mathbb{E}[\phi_{v'} | \mathcal{F}_{\widetilde{B}^c}])] \\ &\quad + \mathbb{E}[(\mathbb{E}[\phi_v | \mathcal{F}_{\widetilde{B}^c}] - \phi_{[v]_\alpha})(\mathbb{E}[\phi_{v'} | \mathcal{F}_{\widetilde{B}^c}] - \phi_{[v']_\alpha})] \ . \end{aligned}$$

The first term is positive by Lemma 5.1. As for the second, Equation (4.4) shows that

$$\mathbb{E}\Big[\left(\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}]-\phi_{[v]_\alpha}\right)\left(\mathbb{E}[\phi_{v'}|\mathcal{F}_{\widetilde{B}^c}]-\phi_{[v']_\alpha}\right)\Big]=O_N(1)\ .$$

This concludes the proof of the lemma. □

Equation (4.2) implies that the free energy of ψ is smaller than the one of $\widetilde{\psi}$ by a standard comparison lemma, see Lemma 5.3 in the Appendix. It remains to prove an upper bound for the free energy of $\widetilde{\psi}$.

Note that the field $\widetilde{\psi}$ is not a GREM *per se* because the variances of $g_B^{(1)}$, $B\in\mathcal{B}_\alpha$, are not the same for every B, as it depends on the distance of B to the boundary. However, the variances of $\phi_{\widetilde{B}}$, $B\in\mathcal{B}_\alpha$, are uniformly bounded by $\frac{\alpha}{\pi}\log N^2+O_N(1)$; indeed

$$\begin{aligned}
\mathbb{E}\left[\phi_{\widetilde{B}}^2\right]&=\mathbb{E}\left[\phi_{v_B}^2\right]-\mathbb{E}\left[(\phi_{v_B}-\phi_{\widetilde{B}})^2\right]\\
&=\mathbb{E}\left[\phi_{v_B}^2\right]-\frac{1-\alpha}{\pi}\log N^2+O_N(1)\\
&\leq\frac{1}{\pi}\log N^2-\frac{1-\alpha}{\pi}\log N^2+O_N(1)=\frac{\alpha}{\pi}\log N^2+O_N(1),
\end{aligned}$$

where we used Lemmas 5.1 and 5.2 in the second line and Lemma 5.2 in the third.

Moreover, note that for $v\in B$,

$$\mathbb{E}[(g_v^{(2)})^2]=\mathbb{E}[\psi_v^2]-\mathbb{E}[(g_B^{(1)})^2]=\sigma_1^2\big(\mathbb{E}[\phi_{[v]_\alpha}^2]-\mathbb{E}[\phi_{\widetilde{B}}^2]\big)+\sigma_2^2\frac{1-\alpha}{\pi}\log N^2-C\sigma_1^2\ .$$

The first term is of order $O_N(1)$ by Equations (4.4) and (4.5). Thus one has

$$\mathbb{E}[(g_v^{(2)})^2]=\sigma_2^2\frac{1-\alpha}{\pi}\log N^2+O_N(1)\ .$$

The important point is that the variance of $g_v^{(2)}$ of $\widetilde{\psi}$ is uniform in v, up to lower order terms. Now consider the 2-level GREM $(\bar{\psi}_v, v\in A_{N,\rho})$

$$\bar{\psi}_v=\bar{g}_B^{(1)}+g_v^{(2)} \tag{4.7}$$

where $(g_v^{(2)}, v\in A_{N,\rho})$ are as before and $(\bar{g}_B^{(1)}, B\in\mathcal{B}_\alpha)$ are i.i.d. Gaussians of variance $\frac{\alpha}{\pi}\log N^2+O_N(1)$. This field differs from $\widetilde{\psi}$ only from the fact that the variance of $\bar{g}_B^{(1)}$ is the same for all B and is the maximal variance of $(g_B^{(1)}, B\in\mathcal{B}_\alpha)$. The calculation of the free energy of $(\bar{\psi}_v, v\in A_{N,\rho})$ is a standard computation and gives the correct upper bound in the statement of Theorem 2.1. (We refer to [9] for the detailed computation of the free energy of the GREM.) The fact that the free energy of $\bar{\psi}$ is larger than the one of $\widetilde{\psi}$ follows from the next lemma showing that the free energy of a hierarchical field is an increasing function of the variance of each point at the first level.

Lemma 4.2. *Consider $N_1,N_2\in\mathbb{N}$. Let $(X_{v_1}^{(1)}, v_1\leq N_1)$ and $(X_{v_1,v_2}^{(2)}; v_1\leq N_1, v_2\leq N_2)$. Consider the Gaussian field of the form*

$$X_v=\sigma_1(v_1)X_{v_1}^{(1)}+\sigma_2X_{v_1,v_2}^{(1)}\ ,\quad v=(v_1,v_2)$$

where $\sigma_2>0$ and $\sigma_1(v_1)>0$, $v_1\leq N_1$, might depend on v_1. Then $\mathbb{E}\left[\log\sum_v e^{\beta X_v}\right]$ is an increasing function in each variable $\sigma_1(v_1)$.

Proof. Direct differentiation gives

$$\frac{\partial}{\partial \sigma_1(v_1)}\mathbb{E}\left[\log \sum_v e^{\beta X_v}\right] = \beta\mathbb{E}\left[\frac{\sum_{v_2} X_{v_1} e^{\beta X_{v_1,v_2}}}{Z_N(\beta)}\right] ,$$

where $Z_N(\beta) = \sum_v e^{\beta X_v}$. Gaussian integration by part then yields

$$\beta\mathbb{E}\left[\frac{\sum_{v_2} e^{X_{v_1}\beta X_{v_1,v_2}}}{\sum_v e^{\beta X_v}}\right] = \beta^2\sigma_1(v_1)\mathbb{E}\left[\frac{\sum_{v_2} e^{\beta X_{v_1,v_2}}}{Z_N(\beta)} - \frac{\sum_{v_2,v_2'} e^{\beta X_{v_1,v_2}} e^{\beta X_{v_1,v_2'}}}{Z_N(\beta)^2}\right] .$$

The right side is clearly positive, hence proving the lemma. □

4.2. **Proof of the lower bound.** Recall the definition of V_N^δ given in the introduction. The two following propositions are used to compute the log-number of high points of the field ψ in V_N^δ. The treatment follows the treatment of Daviaud [15] for the standard GFF. The lower bound for the free energy is then computed using Laplace's method. Define for simplicity $V_{12} := \sigma_1^2\alpha + \sigma_2^2(1-\alpha)$.

Proposition 4.3.

$$\lim_{N\to\infty} \mathbb{P}\left(\max_{v\in V_N^\delta} \psi_v \geq \sqrt{\frac{2}{\pi}}\gamma_{max} \log N^2\right) = 0,$$

where

$$\gamma_{max} = \gamma_{max}(\alpha, \boldsymbol{\sigma}) := \begin{cases} \sqrt{V_{12}}, & \text{if } \sigma_1 \leq \sigma_2, \\ \sigma_1\alpha + \sigma_2(1-\alpha), & \text{if } \sigma_1 \geq \sigma_2. \end{cases}$$

Proof. The case $\sigma_1 \leq \sigma_2$ is direct by a union bound. In the case $\sigma_1 \geq \sigma_2$, note that the field $\widetilde{\psi}$ defined in (4.1) but restricted to V_N^δ is a 2-level GREM with $cN^{2\alpha}$ (for some $c > 0$) Gaussian variables of variance $\frac{\sigma_1^2\alpha}{\pi}\log N^2 + O_N(1)$ at the first level. Indeed, for the field restricted to V_N^δ, the variance of $\mathbb{E}[\phi_{\frac{2}{B}}]$ is $\frac{\sigma_1^2\alpha}{\pi}\log N^2 + O_N(1)$ by Lemma 5.2 since the distance to the boundary is a constant times N. Therefore, by Lemma 5.3 and Equation (4.2), we have

$$\mathbb{P}\left(\max_{v\in V_N^\delta} \psi_v \geq \sqrt{\frac{2}{\pi}}\gamma_{max}\log N^2\right) \leq \mathbb{P}\left(\max_{v\in V_N^\delta} \widetilde{\psi}_v \geq \sqrt{\frac{2}{\pi}}\gamma_{max}\log N^2\right) .$$

The result then follows from the maximal displacement of the 2-level GREM. We refer the reader to Theorem 1.1 in [10] for the details. □

Proposition 4.4. *Let* $\mathcal{H}_N^{\psi,\delta}(\gamma) := \left\{v \in V_N^\delta : \psi_v \geq \sqrt{\frac{2}{\pi}}\gamma \log N^2\right\}$ *be the set of* γ*-high points within* V_N^δ *and define*

$$\text{if } \sigma \leq \sigma_2 \qquad \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma) := 1 - \frac{\gamma^2}{V_{12}};$$

$$\text{if } \sigma \geq \sigma_2 \qquad \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma) := \begin{cases} 1 - \frac{\gamma^2}{V_{12}}, & \text{if } \gamma < \frac{V_{12}}{\sigma_1}, \\ (1-\alpha) - \frac{(\gamma-\sigma_1\alpha)^2}{\sigma_2^2(1-\alpha)}, & \text{if } \gamma \geq \frac{V_{12}}{\sigma_1}. \end{cases}$$

Then, for all $0 < \gamma < \gamma_{max}$*, and for any* $\mathcal{E} < \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma)$*, there exists c such that*

$$\mathbb{P}\left(|\mathcal{H}_N^{\psi,\delta}(\gamma)| \leq N^{2\mathcal{E}}\right) \leq \exp\{-c(\log N)^2\}. \tag{4.8}$$

Proposition 4.4 is obtained by a two-step recursion. Two lemmas are needed. The first is a straightforward generalization of the lower bound in Daviaud's theorem (see Theorem 1.2 in [15] and its proof). For all $0<\alpha<1$, denote by Π_α the centers of the square boxes in $\mathcal{B}_\alpha$ (as defined in Section 4.1) which also belong to V_N^δ.

Lemma 4.5. *Let $\alpha',\alpha''\in(0,1]$ such that $0<\alpha'<\alpha''\leq\alpha$ or $\alpha\leq\alpha'<\alpha''\leq 1$. Denote by σ the parameter σ_1 if $0<\alpha'<\alpha''\leq\alpha$ and by σ the parameter σ_2 if $\alpha\leq\alpha'<\alpha''\leq 1$. Assume that the event*

$$\Xi := \left\{ \#\{v\in\Pi_{\alpha'} : \psi_v(\alpha')\geq \gamma'\sqrt{\frac{2}{\pi}}\log N^2\} \geq N^{\mathcal{E}'} \right\},$$

is such that

$$\mathbb{P}(\Xi^c)\leq \exp\{-c'(\log N)^2\},$$

for some $\gamma'\geq 0$, $\mathcal{E}'>0$ and $c'>0$.

Let

$$\mathcal{E}(\gamma) := \mathcal{E}'+(\alpha''-\alpha')-\frac{(\gamma-\gamma')^2}{\sigma^2(\alpha''-\alpha')}>0.$$

Then, for any γ'' such that $\mathcal{E}(\gamma'')>0$ and any $\mathcal{E}<\mathcal{E}(\gamma'')$, there exists c such that

$$\mathbb{P}\left(\#\{v\in\Pi_{\alpha''} : \psi_v(\alpha'')\geq \gamma''\sqrt{\frac{2}{\pi}}\log N^2\} \leq N^{2\mathcal{E}} \right) \leq \exp\{-c(\log N)^2\}.$$

We stress that γ'' may be such that $\mathcal{E}(\gamma'')<\mathcal{E}'$. The second lemma, which follows, serves as the starting point of the recursion and is proved in [7] (see Lemma 8 in [7]).

Lemma 4.6. *For any α_0 such that $0<\alpha_0<\alpha$, there exists $\mathcal{E}_0=\mathcal{E}_0(\alpha_0)>0$ and $c=c(\alpha_0)$ such that*

$$\mathbb{P}\left(\#\{v\in\Pi_{\alpha_0} : \psi_v(\alpha_0)\geq 0\}\leq N^{\mathcal{E}_0}\right)\leq \exp\{-c(\log N)^2\}.$$

Proof of Proposition 4.4. Let γ such that $0<\gamma<\gamma_{max}$ and choose $\mathcal{E}$ such that $\mathcal{E}<\mathcal{E}^{(\alpha,\sigma)}(\gamma)$. By Lemma 4.6, for $\alpha_0<\alpha$ arbitrarily close to 0, there exists $\mathcal{E}_0=\mathcal{E}_0(\alpha_0)>0$ and $c_0=c_0(\alpha_0)>0$, such that

$$\mathbb{P}\left(\#\{v\in\Pi_{\alpha_0} : \psi_v(\alpha_0)\geq 0\}\leq N^{2\mathcal{E}_0}\right)\leq \exp\{-c_0(\log N)^2\}. \tag{4.9}$$

Moreover, let

$$\mathcal{E}_1(\gamma_1) := \mathcal{E}_0+(\alpha-\alpha_0)-\frac{\gamma_1^2}{\sigma_1^2(\alpha-\alpha_0)}. \tag{4.10}$$

Lemma 4.5 is applied from α_0 to α. For any γ_1 with $\mathcal{E}_1(\gamma_1)>0$ and any $\mathcal{E}_1<\mathcal{E}_1(\gamma_1)$, there exists $c_1>0$ such that

$$\mathbb{P}\left(\#\{v\in\Pi_{\alpha} : \psi_v(\alpha)\geq \gamma_1\sqrt{\frac{2}{\pi}}\log N^2\} \leq N^{2\mathcal{E}_1} \right) \leq \exp\{-c_1(\log N)^2\}.$$

Therefore, Lemma 4.5 can be applied again from α to 1 for any γ_1 with $\mathcal{E}_1(\gamma_1)>0$. Define similarly $\mathcal{E}_2(\gamma_1,\gamma_2):=\mathcal{E}_1(\gamma_1)+(1-\alpha)-(\gamma_2-\gamma_1)^2/\sigma_2^2(1-\alpha)$. Then, for any γ_2 with $\mathcal{E}_2(\gamma_1,\gamma_2)>0$, and $\mathcal{E}_2<\mathcal{E}_2(\gamma_1,\gamma_2)$, there exists $c_2>0$ such that

$$\mathbb{P}\left(\#\{v\in V_N^\delta : \psi_v\geq \gamma_2\sqrt{\frac{2}{\pi}}\log N^2\} \leq N^{2\mathcal{E}_2} \right) \leq \exp\{-c_2(\log N)^2\}. \tag{4.11}$$

Observing that $0 \le \mathcal{E}_0 \le \alpha_0$, Equation (4.8) follows from (4.11) if it is proved that $\lim_{\alpha_0 \to 0} \mathcal{E}_2(\gamma_1,\gamma) = \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma)$ for an appropriate choice of γ_1 (in particular such that $\mathcal{E}_1(\gamma_1) > 0$). It is easily verified that, for a given γ, the quantity $\mathcal{E}_2(\gamma_1,\gamma)$ is maximized at $\gamma_1^* = \gamma\sigma_1^2(\alpha-\alpha_0)/(V_{12}-\sigma_1^2\alpha_0)$. Plugging these back in (4.10) shows that $\mathcal{E}_1(\gamma_1^*) > 0$ provided that $\gamma < V_{12}/\sigma_1 =: \gamma_{crit}$, with α_0 small enough (depending on γ). Furthermore, since $\mathcal{E}_2(\gamma_1^*,\gamma) = \mathcal{E}_0 + (1-\alpha_0) - \gamma^2/(V_{12}-\sigma_1^2\alpha_0)$, we obtain $\lim_{\alpha_0\to 0}\mathcal{E}_2(\gamma_1^*,\gamma) = \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma)$, which concludes the proof in the case $0<\gamma<\gamma_{crit}$.

If $\gamma_{crit} \le \gamma < \gamma_{max}$, the condition $\mathcal{E}_1(\gamma_1^*) > 0$ is violated as α_0 goes to zero. However, the previous arguments can easily be adapted and we refer to subsection 3.1.2 in [4] for more details. □

Proof of the lower bound of Theorem 2.1. We will prove that for any $\nu > 0$

$$\mathbb{P}\left(f_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta) \le f^{(\alpha,\boldsymbol{\sigma})}(\beta) - \nu\right) \longrightarrow 0, \qquad N \to 0.$$

Define $\gamma_i := i\gamma_{\max}/M$ for $0 \le i \le M$ (M will be chosen large enough). Notice that Proposition 4.3, Proposition 4.4 and the symmetry property of centered Gaussian random variables imply that the event

$$B_{N,M,\nu} \;:=\; \bigcap_{i=0}^{M-1}\left\{|\mathcal{H}_N^{\psi,\delta}(\gamma_i)| \ge N^{2\mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma_i)-\nu/3}\right\}\bigcap\left\{\max_{v\in V_N^\delta}|\psi_v| \le \sqrt{\frac{2}{\pi}}\gamma_{\max}\log N^2\right\}$$

satisfies

$$\mathbb{P}(B_{N,M,\nu}) \longrightarrow 1, \qquad N\to\infty,$$

for all $M \in \mathbb{N}^*$ and all $\nu > 0$. Then, observe that on $B_{N,M,\nu}$

$$\begin{aligned}
Z_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta) &\ge \sum_{v\in V_N^\delta} \mathrm{e}^{\beta\psi_v} \ge \sum_{i=1}^{M}(|\mathcal{H}_N^{\psi,\delta}(\gamma_{i-1})| - |\mathcal{H}_N^{\psi,\delta}(\gamma_i)|)N^{2\sqrt{\frac{2}{\pi}}\gamma_{i-1}\beta}\\
&= |\mathcal{H}_N^{\psi,\delta}(0)| + \left(2\sqrt{\frac{2}{\pi}}\frac{\gamma_{\max}}{M}\beta\log N\right)\int_1^M |\mathcal{H}_N^{\psi,\delta}(\frac{\lfloor u\rfloor\gamma_{\max}}{M})|N^{2\sqrt{\frac{2}{\pi}}\frac{u-1}{M}\gamma_{\max}\beta}du\\
&\ge \left(2\sqrt{\frac{2}{\pi}}\frac{\gamma_{\max}}{M}\beta\log N\right)\sum_{i=1}^{M-1}|\mathcal{H}_N^{\psi,\delta}(\gamma_i)|N^{2\sqrt{\frac{2}{\pi}}\gamma_{i-1}\beta},
\end{aligned}$$

where we used Abel's summation by parts formula. Writing $\gamma_{i-1} = \gamma_i - \gamma_{\max}/M$ and $P_\beta(\gamma) := \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma) + \sqrt{\frac{2}{\pi}}\beta\gamma$, we get on $B_{N,M,\nu}$

$$f_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta) \ge \frac{1}{\log N^2}\log\left(\sum_{i=1}^{M-1}N^{2P_\beta(\gamma_i)}\right) - \frac{\nu}{6} - \frac{\sqrt{\frac{2}{\pi}}\gamma_{\max}\beta}{M} + o_N(1)\ . \tag{4.12}$$

Using the expression of $\mathcal{E}^{(\alpha,\boldsymbol{\sigma})}$ in Proposition 4.4 on the different intervals, it is easily checked by differentiation that $\max_{\gamma\in[0,\gamma_{\max}]}P_\beta(\gamma) = f^{(\alpha,\boldsymbol{\sigma})}(\beta)$. Furthermore, the continuity of $\gamma \mapsto P_\beta(\gamma)$ on $[0,\gamma_{\max}]$ yields

$$\max_{1\le i\le M-1}P_\beta(\gamma_i) \longrightarrow \max_{\gamma\in[0,\gamma_{\max}]}P_\beta(\gamma) = f^{(\alpha,\boldsymbol{\sigma})}(\beta), \qquad M\to\infty.$$

Therefore, choosing M large enough and applying Laplace's method in (4.12) yield the result. □

5. Appendix

The conditional expectation of the GFF has nice features such as the Markov property, see e.g. Theorems 1.2.1 and 1.2.2 in [21] for a general statement on Markov fields constructed from symmetric Markov processes.

Lemma 5.1. *Let $B \subset A$ be subsets of $\mathbb{Z}^2$. Let $(\phi_v, v \in A)$ be a GFF on A. Then*

$$\mathbb{E}[\phi_v | \mathcal{F}_{B^c}] = \mathbb{E}[\phi_v | \mathcal{F}_{\partial B}], \qquad \forall v \in B,$$

and

$$(\phi_v - \mathbb{E}[\phi_v | \mathcal{F}_{\partial B}], v \in B)$$

has the law of a GFF on B. Moreover, if P_v is the law of a simple random walk starting at v and τ_B is the first exit time of B, we have

$$\mathbb{E}[\phi_v | \mathcal{F}_{\partial B}] = \sum_{u \in \partial B} P_v(S_{\tau_B} = u) \ \phi_u \ .$$

The following estimate on the Green function can be found as Lemma 2.2 in [18] and is a combination of Proposition 4.6.2 and Theorem 4.4.4 in [28].

Lemma 5.2. *There exists a function $a : \mathbb{Z}^2 \times \mathbb{Z}^2 \mapsto [0, \infty)$ of the form*

$$a(v, v') = \frac{2}{\pi} \log \|v - v'\| + \frac{2\gamma_0 \log 8}{\pi} + O(\|v - v'\|^{-2})$$

(where γ_0 denotes the Euler's constant) such that $a(v, v) = 0$ and

$$G_A(v, v') = E_v \left[a(v', S_{\tau_A})\right] - a(v, v') \ .$$

Slepian's comparison lemma can be found in [27] and in [26] for the result on log-partition function.

Lemma 5.3. *Let $(X_1, \cdots, X_N)$ and $(Y_1, \cdots, Y_N)$ be two centered Gaussian vectors in N variables such that*

$$\mathbb{E}[X_i^2] = \mathbb{E}[Y_i^2] \ \forall i, \qquad \mathbb{E}[X_i X_j] \geq \mathbb{E}[Y_i Y_j] \ \forall i \neq j \ .$$

Then for all $\beta > 0$

$$\mathbb{E}\left[\log \sum_{i=1}^{N} e^{\beta X_i}\right] \leq \mathbb{E}\left[\log \sum_{i=1}^{N} e^{\beta Y_i}\right] \ ,$$

and for all $\lambda > 0$,

$$\mathbb{P}\left(\max_{i=1,\ldots,N} X_i > \lambda\right) \leq \mathbb{P}\left(\max_{i=1,\ldots,N} Y_i > \lambda\right) \ .$$

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